OUT–4102–67
hep-th/9612013
30 November 1996

Compact analytical form for non-zeta terms in critical exponents at order $1/N^3$

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Abstract We simplify, to a single integral of dilogarithms, the least tractable $O(1/N^3)$ contribution to the large-N critical exponent η of the non-linear σ-model, and hence ϕ^4-theory, for any spacetime dimensionality, D. It is the sole generator of irreducible multiple zeta values in ε-expansions with $D = 2 - 2\varepsilon$, for the σ-model, and $D = 4 - 2\varepsilon$, for ϕ^4-theory. In both cases we confirm results of Broadhurst, Gracey and Kreimer (BGK) that relate knots to counterterms. The new compact form is much simpler than that of BGK. It enables us to develop 8 new terms in the ε-expansion with $D = 3 - 2\varepsilon$. These involve *alternating* Euler sums, for which the basis of irreducibles is larger. We conclude that massless Feynman diagrams in odd spacetime dimensions share the greater transcendental complexity of massive diagrams in even dimensions, such as those contributing to the electron's magnetic moment and the electroweak ρ-parameter. Consequences for the perturbative sector of Chern-Simons theory are discussed.

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1. Introduction

Since the pioneering work of the St Petersburg group [1, 2], exploiting conformal invariance [3] of critical phenomena, it was known that the $\mathrm{O}(1/N^3)$ term η_3 in the large-N critical exponent η of the non-linear σ-model, or equivalently ϕ^4-theory, in any number $D \equiv 2\mu$ of spacetime dimensions, derives its maximal complexity from a single Feynman integral $I(\mu)$, defined below. The situation is thoroughly reviewed by Broadhurst, Gracey and Kreimer in [4] (hereafter BGK), who first showed that the ε-expansions of $I(1-\varepsilon)$ and $I(2-\varepsilon)$, for the σ-model and ϕ^4-theory, entail double Euler sums [5], of the type $\zeta(s,t)=\sum_{m>n>0} m^{-s}n^{-t}$. The majority of such sums are conjectured [6, 7, 8, 9, 10, 11, 12] to be irreducible to the single sums $\zeta(r)=\sum_{n>0} n^{-r}$, when s and t have the same parity, the weight $s+t$ exceeds 6, and $s>t>1$. Prior to BGK, the first double-sum irreducible, conjectured [13] to be at weight 8, had not been detected by large-N studies [14, 15, 16].

Thus $I(\mu)$ provides important information [4] on the mapping [17, 18] between positive knots [19, 20] and transcendental numbers [8, 21, 22], realized by the counterterms [23, 24, 25, 26, 27] of perturbative quantum field theory, in even numbers of spacetime dimensions, and most pertinently in the four dimensions where particle physics is studied experimentally. This connection arises from the skeining [28] of link diagrams [17] that encode momentum flow in Feynman diagrams. The field-theoretic connection between $\zeta(2n+1)$ and the 2-braid torus knot [28] $(2n+1,2)$ is now well understood [17, 18, 26]. Thanks to [2], BGK were able to explore the field-theoretic connection between 3-braid knots and irreducible double Euler sums, at loop orders much higher than the 7 loops achieved in [23], and hence for knots with many crossings and Euler sums of large weights.

The purpose of the present paper is to give a representation of $I(\mu)$ that is considerably simpler than that achieved by BGK, and to exploit it by addressing a further question in knot/number/field theory [20]: what is the character of the transcendentals that emerge from massless, single-scale Feynman diagrams in *odd* numbers of dimensions?

These possibilities arise from the work of one us (AVK) in [29], following communication from the other (DJB) that BGK had succeeded in reducing a large class of Feynman integrals, including $I(\mu)$, to ${}_3F_2$ series, of the type first revealed in [30] and intensively studied in [31]. As a result, an alternative route to ${}_3F_2$ series was found in [29], via Gegenbauer-polynomial techniques [32]. While such techniques are sometimes less efficient than the use of recurrence relations [4, 31], and require considerable ingenuity to convert [29] Gegenbauer double series to ${}_3F_2$ series, they have facilitated the current project of studying the ε-expansion of $I(\frac{3}{2}-\varepsilon)$, for which only the leading [2] term, $I(\frac{3}{2})=-7\zeta(3)/2\zeta(2)+2\ln 2$, was previously known. We shall show that higher terms in this ε-expansion, involving more loops in the regularization of three-dimensional Feynman diagrams, entail irreducible *alternating* Euler sums. In contrast, non-alternating sums, which we call multiple zeta values (MZVs) [7, 11, 19], emerge in even dimensions. Massless multiloop diagrams in odd dimensions thus have greater analytical complexity than those in even dimensions. Indeed, single-scale massless diagrams in three dimensions appear to have the same character as single-scale massive [22, 31] diagrams in four dimensions.

In Section 2 we recall the definition [2] of $I(\mu)$ and give the new result for it. A derivation is sketched in Section 3. Sections 4 and 5 show how MZVs and alternating sums emerge in even and odd dimensions, respectively. Section 6 gives our conclusions.

2. New compact result for $I(\mu)$

The parameter $\mu \equiv D/2$, used in [1, 2], is less convenient than $\lambda \equiv \mu - 1$, which arises as the exponent in the bare propagator, $[1/(x-y)^2]^\lambda \equiv 1/(x-y)^{2\lambda}$, of a scalar particle in D-dimensional configuration space. Our aim is to expand $I(\mu) = I(\lambda+1)$ around $\lambda = 0$, and $\lambda = \frac{1}{2}$, corresponding to $D = 2$ and $D = 3$, respectively. Expansions around larger integer values of D may then be achieved by a recurrence relation [4, 16] that increases the dimensionality by two units.

The definition of I is [2]

$$I(\lambda+1) = \frac{\mathrm{d}}{\mathrm{d}\Delta} \ln \Pi(\lambda, \Delta)\bigg|_{\Delta=0}, \tag{1}$$

where

$$\Pi(\lambda, \Delta) = \frac{x^{2(\lambda+\Delta)}}{\pi^D} \int\int \frac{\mathrm{d}^D y \mathrm{d}^D z}{y^2 z^2 (x-y)^{2\lambda}(x-z)^{2\lambda}(y-z)^{2(\lambda+\Delta)}} \tag{2}$$

is a two-loop two-point integral, with three dressed propagators, made dimensionless by the appropriate power of x^2.

The new result, obtained by AVK, is

$$I(\lambda+1) = \Psi(1) - \Psi(1-\lambda) + \frac{\Phi(\lambda) - \frac{1}{3}\Psi''(\lambda) - \frac{7}{24}\Psi''(1)}{\Psi'(1) - \Psi'(\lambda)}, \tag{3}$$

where

$$\Phi(\lambda) = 4\int_0^1 dx \frac{x^{2\lambda-1}}{1-x^2} \left\{\mathrm{Li}_2(-x) - \mathrm{Li}_2(-1)\right\}, \tag{4}$$

with $\Psi(x) = \Gamma'(x)/\Gamma(x)$ and $\mathrm{Li}_2(x) = \sum_{n>0} x^n/n^2$.

3. Method of derivation

We begin by considering a more convenient Feynman integral, namely

$$J(\lambda, \Delta) = \frac{x^{2(2\lambda+\Delta-1)}}{\pi^D} \int\int \frac{\mathrm{d}^D y \mathrm{d}^D z}{y^{2\lambda} z^{2\lambda} (x-y)^{2\lambda}(x-z)^{2\lambda}(y-z)^{2(1+\Delta)}}, \tag{5}$$

with a dressing only on the internal propagator. It was studied in [29], with the result

$$J(\lambda, \Delta) = \frac{2}{\lambda+\Delta-1} \frac{\Gamma(2\lambda+\Delta-1)}{\Gamma^2(\lambda)\Gamma(2\lambda)} \frac{A(\lambda,\Delta) - B(\lambda,\Delta)}{\Delta}, \tag{6}$$

where

$$A(\lambda, \Delta) = \sum_{n=0}^{\infty} \frac{1}{n+\lambda+\Delta} \frac{\Gamma(n+2\lambda)}{\Gamma(n+2\lambda+\Delta)}, \tag{7}$$

$$B(\lambda, \Delta) = \frac{\Gamma(\lambda-\Delta)}{\Gamma(\lambda)} \frac{\pi}{\tan(\pi\Delta)}. \tag{8}$$

It is convenient to split $A = A_1 + A_2$ into an easy and a difficult part:

$$A_1(\lambda,\Delta) = \sum_{n=0}^{\infty} \frac{1}{n+\lambda+\Delta} \frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda+\Delta)} = \frac{1}{\Delta} \frac{\Gamma(\lambda)}{\Gamma(\lambda+\Delta)}\,, \tag{9}$$

$$A_2(\lambda,\Delta) = \sum_{n=0}^{\infty} \frac{1}{n+\lambda+\Delta} \left\{ \frac{\Gamma(n+2\lambda)}{\Gamma(n+2\lambda+\Delta)} - \frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda+\Delta)} \right\}. \tag{10}$$

Only the first two terms of $A_2(\lambda,\Delta) = A_{2,1}(\lambda)\Delta + \frac{1}{2}A_{2,2}(\lambda)\Delta^2 + \mathrm{O}(\Delta^3)$ are needed. By laborious procedures, whose detail would be inappropriate here, these Taylor coefficients were obtained as

$$A_{2,1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Psi(n+\lambda) - \Psi(n+2\lambda)}{n+\lambda} = -\frac{\Psi'(1) + \Psi'(\lambda)}{2}\,, \tag{11}$$

$$A_{2,2}(\lambda) = 3\Phi(\lambda) - \Psi''(\lambda) - \tfrac{7}{8}\Psi''(1) + \Psi(\lambda)\left\{\Psi'(1) - \Psi'(\lambda)\right\}\,, \tag{12}$$

where Φ is defined in (4), and is the sole term that is not reducible to polygammas. Hence we are able to expand (5) to $\mathrm{O}(\Delta)$.

The transformation labelled "$\rightarrow$" in [1] then yields

$$\Pi(\lambda,\Delta) = \frac{\Gamma^2(\lambda)\Gamma(\lambda+\Delta)\Gamma(2-\Delta-\lambda)}{\Gamma(1-\Delta)\Gamma(2\lambda+\Delta-1)}\, J(\lambda,\Delta)\,, \tag{13}$$

whose $\mathrm{O}(\Delta)$ term is needed in (1). Collecting terms, we obtain the advertised result (3).

4. MZVs in even dimensions

It was found in [4, 16] that $I(\lambda+1)$ is related to $I(\lambda)$ by a rather complex recurrence relation. The origin of this is immediately apparent from (4), which exposes the integral at the heart of the intractability of $I(\lambda+1)$. By expanding $\mathrm{Li}_2(-x) = \sum_{k>0}(-x)^k/k^2$, one obtains the recurrence relation

$$\Phi(\lambda) - \Phi(\lambda+1) = 4\sum_{k>0} \frac{(-1)^k}{k^2} \left\{ \frac{1}{2\lambda+k} - \frac{1}{2\lambda} \right\} = \frac{\Psi(2\lambda+1) - \Psi(\lambda+1)}{\lambda^2}\,, \tag{14}$$

which shifts D by 2. Its solution is

$$\Phi(\lambda) = 4\sum_{k>0} \frac{(-1)^k}{k^2} \sum_{n\geq 0} \left\{ \frac{1}{2\lambda+k+n} - \frac{1}{2\lambda+n} \right\} \frac{1+(-1)^n}{2}\,, \tag{15}$$

which may be obtained directly from (4) by expanding both $1/(1-x^2)$ and $\mathrm{Li}_2(-x)$.

It is apparent from (15) that ε-expansions in $D = 2 - 2\varepsilon$ and $D = 3 - 2\varepsilon$ dimensions may be obtained in terms of alternating Euler sums. The results are

$$\Phi(-\varepsilon) + \frac{\zeta(2)}{\varepsilon} = -3\zeta(2)\left\{\ln 2 + \Psi(1-\varepsilon) - \Psi(1-2\varepsilon)\right\} - 2\sum_{r>0}(2\varepsilon)^{r-1}T_+(2,r)\,, \tag{16}$$

$$\Phi(\tfrac{1}{2} - \varepsilon) = 3\zeta(2)\left\{\ln 2 + \Psi(1-\varepsilon) - \Psi(1-2\varepsilon)\right\} - 2\sum_{r>0}(2\varepsilon)^{r-1}T_-(2,r)\,, \tag{17}$$

where

$$T_{\pm}(s,t)=\sum_{m>n>0}\frac{(-1)^m\pm(-1)^n}{m^s n^t}\,. \tag{18}$$

For $D=4-2\varepsilon$ one has merely to set $\lambda=-\varepsilon$ in (14) and use (16).

The simplicity of (16,17) relies on the use of *alternating* sums in (18). However, $\Phi(-\varepsilon)$ should be expandable in terms of non-alternating sums (i.e. MZVs) since that was the finding of BGK, where a result was obtained for $I(1-\varepsilon)$ in terms of the double sum

$$S_+(\varepsilon)\equiv\sum_{m>n>0}\frac{\varepsilon^3}{(m+\varepsilon)^2(n-\varepsilon)}\;+\;(\varepsilon\to-\varepsilon)\;=\sum_{s,t>0}(s-1)\zeta(s,t)\left\{(-1)^s\pm(-1)^t\right\}\varepsilon^{s+t} \tag{19}$$

and the polygammas $\psi_p^{(n)}\equiv\{\partial/\partial p\}^{n+1}\ln\Gamma(1+p\,\varepsilon)=\varepsilon^{n+1}\Psi^{(n)}(1+p\,\varepsilon)$.

Recasting the result of BGK in terms of (3), we find that

$$\begin{aligned}\varepsilon^3\Phi(-\varepsilon)\;&=\;\tfrac12 S_+(\varepsilon)+\tfrac18(3\psi_1-7\psi_{-1}-2\psi_2+6\psi_{-2}+6\psi_0'-3\psi_1'-5\psi_{-1}')\\&\quad-\tfrac{1}{16}(\psi_1''-3\psi_{-1}'')-\tfrac32\psi_0'(\psi_1-\psi_{-1}-\psi_2+\psi_{-2})\\&\quad+\tfrac14\psi_1'(\psi_1+\psi_{-1}-\psi_2-\psi_{-2})+\tfrac14\psi_{-1}'(\psi_1-3\psi_{-1}-\psi_2+3\psi_{-2})\,,\end{aligned} \tag{20}$$

which we have verified, up to terms of $\mathrm{O}(\varepsilon^{19})$, using a suitable basis [22] for all double Euler sums up to weight 19, obtained by methods developed in [8] and augmented in [22]. A convenient $\mathbf{Q}$-basis for double sums, which is conjectured [22] to be minimal, is formed by $\ln 2$, π^2, $\{\zeta(2a{+}1)\mid a>0\}$ and the alternating double sums $\{U(2a{+}1,2b{+}1)\mid a>b\geq 0\}$, of the form [13] $U(s,t)=\sum_{m>n}(-1)^{m+n}m^{-s}n^{-t}$. All 3698 convergent double Euler sums with weights up to 44 have been expressed in this basis. Moreover high-precision lattice methods [22] reveal no rational relations between the basis elements. It was thus a simple matter of programming to demonstrate the agreement of (16) with (20), up to terms of $\mathrm{O}(\varepsilon^{19})$ in (20).

The circumstance that (16) yields only MZVs may be restated, using the identity

$$\zeta(s,t)+U(s,t)+T_+(s,t)=2^{2-s-t}\zeta(s,t)\,, \tag{21}$$

which is obtained by retaining only even values of m and n in $\sum_{m>n>0}m^{-s}n^{-t}$. From (21) it follows that $T_+(2,r)$, in (16), is expressible in terms of MZVs if and only if $U(2,r)$ is so expressible. AVK has devised an elementary proof of the latter proposition.

Thus it may be seen that the simplicity of (3,16) is won at some price: it disguises the MZV-content of the even-dimensional case, discovered by BGK. However, there was a consequent unexpected gift from field theory to number theory. From the equivalence of the present result for $I(\mu)$ with that of BGK, one of us (DJB) inferred a compact single formula (subsequently proven, from first principles, by Roland Girgensohn) that gives *all* the reductions of odd-weight double sums to single sums and their products. The four possible choices of sign were not treated on an equal footing in [8], where the reduction of alternating sums was left in a 'somewhat more implicit state', compared with the non-alternating case. The reader is referred to a recent compendium [9] of results for Euler sums, where this field-theory byproduct appears as Equation (74), in a notation which unifies all 8 cases that result from the four possible choices of sign, and the two possible choices of the opposite parities of the exponents in odd-weight double sums.

5. Irreducible alternating sums in odd dimensions

For expansion (17), in $D = 3 - 2\varepsilon$ dimensions, MZVs are insufficient. The odd powers of ε in (17) entail $T_-(2, 2r)$, which is a genuinely new double-sum transcendental. However, study of all the cases with weights up to 44 reveals that it may always be expressed in terms of MZVs and the single alternating sum $U(2r+1, 1)$, with a coefficient that follows a regular pattern. Specifically, we obtained the following ε-expansion for $D = 3 - 2\varepsilon$:

$$I(\tfrac{3}{2} - \varepsilon) = \frac{-7\zeta(3) + 4\zeta(2)\ln 2 + \sum_{n>3} Y_n \varepsilon^{n-3}}{2\zeta(2) + \sum_{n>2}(n-1)(2^n - 1)\zeta(n)\varepsilon^{n-2}}\,, \tag{22}$$

where only the leading [2] term, $I(\frac{3}{2}) = -7\zeta(3)/2\zeta(2) + 2\ln 2$, was previously known. Developing the numerator to weight 11, we obtain

$$\begin{aligned}
Y_4 &= -\tfrac{53}{2}\zeta(4) + 16U(3,1)\\
Y_5 &= -42\zeta(3)\zeta(2) - \tfrac{217}{2}\zeta(5) + 90\zeta(4)\ln 2\\
Y_6 &= -456\zeta(6) + 29\zeta^2(3) + 128U(5,1)\\
Y_7 &= 7\zeta(3)\zeta(4) - 434\zeta(5)\zeta(2) - \tfrac{3937}{4}\zeta(7) + 630\zeta(6)\ln 2\\
Y_8 &= -153\zeta(6,2) - \tfrac{18321}{4}\zeta(8) + 822\zeta(5)\zeta(3) + 768U(7,1)\\
Y_9 &= 889\zeta(3)\zeta(6) - 1333\zeta(5)\zeta(4) - 2794\zeta(7)\zeta(2) - \tfrac{20951}{3}\zeta(9) + 3570\zeta(8)\ln 2\\
Y_{10} &= -591\zeta(8,2) - \tfrac{64265}{2}\zeta(10) + 2176\zeta^2(5) + 4340\zeta(7)\zeta(3) + 4096U(9,1)\\
Y_{11} &= 7147\zeta(3)\zeta(8) - 1891\zeta(5)\zeta(6) - 11049\zeta(7)\zeta(4) - 15330\zeta(9)\zeta(2)\\
&\quad - \tfrac{173995}{4}\zeta(11) + 18414\zeta(10)\ln 2
\end{aligned} \tag{23}$$

with $U(2s-1, 1)$ appearing in Y_{2s}, with coefficient $(s-1)4^s$, and $\zeta(2s)\ln 2$ in Y_{2s+1}, with coefficient $2(2s-1)(4^s - 1)$, for $s > 1$. All other contributions are MZVs, though some are irreducible to single sums, as in even dimensions.

We conclude that the appearance of $\ln 2$ in the strictly three-dimensional result of [2], was merely the opening of the floodgates to further non-MZV terms. At weight 4, one encounters [22]:

$$U(3,1) \equiv \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \tfrac{1}{2}\zeta(4) - 2\left\{\mathrm{Li}_4(\tfrac{1}{2}) + \tfrac{1}{24}\ln^2 2\left(\ln^2 2 - \pi^2\right)\right\}\,, \tag{24}$$

where the non-MZV polylogarithm $\mathrm{Li}_4(\frac{1}{2})$ occurs with precisely the same [22] combination of $(\ln 2)^4$ and $(\pi \ln 2)^2$ as in the massive four-dimensional three-loop results for the anomalous magnetic moment of the electron [33] and the ρ-parameter [34] of electroweak theory. Only the combinations of $U(3,1)$ with $\zeta(4) = \frac{\pi^4}{90}$ differ from that in Y_4 of (23). This is a remarkable circumstance, giving concrete support to the idea, motivated by hypergeometric analysis [22, 31], that massless diagrams in odd dimensions lead to results that are transcendentally more complex than those for massless even-dimensional diagrams, and are closely akin to *massive* even-dimensional results. Nor do simple polylogarithms exhaust the novelty; at weight 6 there is no [8, 22] known integer relation between $U(5,1)$, $\mathrm{Li}_6(\frac{1}{2})$ and simpler polylogs.

6. Conclusions

In conclusion, we have obtained and exploited a remarkably simple representation (3,4) for the least tractable integral $I(\mu)$ in the large-N critical exponent η [2] at order $1/N^3$. The even-dimensional ε-expansions of BGK [4] were confirmed up to weight 19, making the probability of error in the new result (3), or in the BGK result, negligible. Checking their equivalence had a beneficial spin-off for number theory [9]. It is remarkable how closely the massless three-dimensional results mimic massive four-dimensional results [33, 34]. The origin of this is clear: alternating Euler sums are the common new ingredient, and (24) is the only [22] possible irreducible depth-two interloper at weight 4, if Euler sums exhaust the transcendentals from single-scale diagrams, at this weight.

A large questions remains: are alternating Euler sums the numbers assigned to knots via counterterms of odd-dimensional field theories? It must be emphasized that we do not yet know the answer. There are manifold successes [4, 19, 23, 24, 25, 26, 27] of using knots [17, 18, 20] to relate the transcendentality content of even-dimensional counterterms to the skeining [28] of link diagrams that encode momentum flow. Very recent work [35, 36] suggests an underlying weight system associated with four-term [37] relations. Yet it is vital to remember that all this was achieved by the study of renormalizable field theories in their critical dimensions, which were *even* in the cases studied so far. The all-order results of [2], for critical exponents, have predictive content for the perturbative sectors of the σ-model and ϕ^4-theory at their critical dimensions of $D = 2$ and $D = 4$, respectively, where counterterms, from nullified diagrams, are the simplest possible field-theoretic constructs. Until one studies a theory whose critical dimension is $D = 3$, one has no right to associate knots with alternating Euler sums, via counterterms.

Surprisingly, there appears to be no literature on ϕ^6-theory, in its critical dimension $D = 3$, beyond the two-loop [38] level. Two-loop counterterms are trivial from the point view of knot theory [17, 18], as the trefoil knot first occurs at three loops. Gratifyingly, they also appear to be trivial from the point of view of number theory, with a rational two-loop beta-function in [38], agreeing with the expectations of [17, 18]. In some renormalization schemes, subdivergences [26] may generate π^2-terms [39] in anomalous dimensions, at $D = 3$, corresponding to framing dependence in knot theory [20]. There is a four-loop [40] analysis of diagrams that are logarithmically divergent for $D = 3$, and this indeed yields π^2 terms. In even dimensions [26], by contrast, the first framing dependence shows up at the level of $\zeta(4) = \pi^4/90$, in the scheme-dependence of anomalous dimensions. We commend study of ϕ^6-theory at the three-loop level, and preferably beyond. It may provide the first non-trivial results on the mapping from knots to transcendental numbers via the counterterms of a field theory whose critical dimension is odd.

In the meantime, our ε-expansion (22,23), for $D = 3 - 2\varepsilon$, fully confirms the hypergeometric expectation of [31] that the transcendental complexity of massless Feynman integrals in three dimensions is comparable to that of massive [33, 34] diagrams in four dimensions and hence entails alternating [22] Euler sums. This is sobering news for colleagues seeking to push back the computational frontier in the perturbative sector [38] of Chern-Simons theory. At present this lags far behind the 7-loop [23] level, achieved for ϕ^4-theory with $D = 4$, in an analysis that spectacularly confirmed Kreimer's predictions [17, 18] for the fascinating nexus of knot/number/field theory [20], whose study has advanced with great rapidity [4, 7, 9, 10, 11, 12, 19, 22, 26, 27, 35, 36] in recent months.

Acknowledgements

AVK is grateful to Prof. A.N. Vasil'ev, for stimulating the quest for (3), and also to Prof. D.I. Kazakov, Dr. N.A. Kivel, and Dr. A.S. Stepanenko, for discussion. DJB thanks Jon Borwein, David Bradley, Roland Girgensohn and Don Zagier, for discussions on number theory, Bob Delbourgo and John Gracey, for discussions on field theory, and Dirk Kreimer, for knotting the two together during a collaboration on [4, 19, 36] at the University of Tasmania, generously hosted by Bob Delbourgo in July and August.

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